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MAXIMIZING THE MINIMUM SOURCE-SINK PATH SUBJECT TO A BUDGET CONSTRAINT: ANOTHER VIEW OF THE MINIMUM COST FLOW ROUTINE

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Consider a network in which each arc has a traversal time and a cost for increasing its traversal time. If a fixed budget is available for expenditure on arcs of the network, how does one allocate this budget among the arcs so as to maximize the length of a shortest path from source to sink? It is shown that the minimum cost flow algorithm can be used to solve this problem parametrically in the budget.

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MAXIMIZING THE MINIMUM SOURCE-SINK PATH SUBJECT TO A BUDGET CONSTRAINT: ANOTHER VIEW OF THE MINIMUM COST FLOW ROUTINE

by

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1. <u>Introduction</u>. Let $G = [N;A]$ be a network with node-set N and arc-set $A \subseteq N \times N$ having source $s \in N$ and sink $t \in N$. Suppose that each arc $(x,y) \in A$ has a positive integral traversal time $a(x,y)$ and also a positive integral cost $c(x,y)$ for increasing the traversal time of this arc, i.e., the cost of increasing the traversal time of arc (x,y) from $a(x,y)$ to $a(x,y) + \gamma(x,y)$ is $c(x,y)\gamma(x,y)$. The question we pose and answer is the following: If a fixed budget b is available for expenditure on arcs of the network, how does one allocate the budget b among the arcs in such a way that the shortest (least time) directed path from s to t is made as large as possible? It is easy to formulate this problem as a linear program, in much the same way as was done in [2] for the problem of optimally increasing the flow-capacity of a network, relative to a designated source and sink, subject to a budget constraint. For the problem at hand, it turns out that the minimum cost flow algorithm of [1, Chap. III, §3], when interpreted appropriately, directly solves the problem parametrically in b. Thus this problem provides another viewpoint on computing minimum cost source to sink flows in a capacity-constrained network.

While we assume that we are dealing with a directed network and directed source-sink paths, this is merely a convenience. Undirected (or mixed) networks can be handled by the usual device of passing to an equivalent directed network [1].

Various practical interpretations can be given for this problem. It can be viewed as an interdiction model of transportation networks, for example. For undirected networks, another physical interpretation is the following. Suppose we have a string-model of the network, where there are $c(x,y)$ strings, each of length $a(x,y)$, joining x and y. We also have at hand an additional piece of string of length b and a pair of scissors. We are allowed to cut up this string into pieces in any way we like, cut the pieces of string in the network and "splice in" the additional pieces as we please, the objective being the following: when we take the source-node in one hand and the sink-node in the other, we want to be able to pull them apart as far as the extra string of length b will permit.

2. Linear programming formulation. Using well-known results about shortest paths [1], we can formulate the problem posed in Section 1 as follows. Associate with each node $x \in N$ a (variable) potential $\pi(x)$. We then want to solve the linear program

$$\text{maximize} \quad \pi(t) - \pi(s) \tag{2.1}$$

subject to the constraints

$$\pi(y) - \pi(x) - \gamma(x,y) \leq a(x,y), \quad (x,y) \in A, \tag{2.2}$$

$$\gamma(x,y) \geq 0, \quad (x,y) \in A, \tag{2.3}$$

$$\sum_A c(x,y)\gamma(x,y) \leq b. \tag{2.4}$$

Here $a(x,y)$, $c(x,y)$ are given positive integers, b is a given nonnegative number, and $\gamma(x,y)$, $\pi(x)$ are variables whose values are to be determined.

The linear program dual to (2.1)-(2.4) can now be written down. Assign dual variables $g(x,y)$, all $(x,y) \in A$, to the constraints (2.2), and a dual variable λ to constraint (2.4). The program dual to (2.1)-(2.4) is:

$$\text{minimize} \quad \sum_A a(x,y)g(x,y) + \lambda b \tag{2.5}$$

subject to the constraints

$$g(x,N) - g(N,x) = \begin{cases} 1, & x = s, \\ -1, & x = t, \\ 0, & \text{otherwise}, \end{cases} \tag{2.6}$$

$$g(x,y) \leq \lambda c(x,y), \quad (x,y) \in A, \tag{2.7}$$

$$g(x,y) \geq 0, \quad (x,y) \in A \tag{2.8}$$

$$\lambda \geq 0. \tag{2.9}$$

(In (2.6),

$$g(x,N) = \sum_{\{y \in N:\ (x,y) \in A\}} g(x,y),$$

$$g(N,x) = \sum_{\{y \in N:\ (y,x) \in A\}} g(y,x).)$$

Thus if we knew λ, we would be seeking a least cost flow g of amount 1 from s to t through the network $G = [N;A]$, where arc (x,y) has unit flow cost $a(x,y)$ and capacity $\lambda c(x,y)$. Hence one could solve the problem, for fixed b, by "searching on λ". There is, however, no need to do this. A better way is to solve the problem parametrically in b, starting with $b = 0$, for which the solution is obvious: take $\lambda = 1$, say, and send one unit along a shortest (least cost) directed path from s to t. This corresponds, in the primal problem (2.1)-(2.4), to taking $\gamma(x,y) = 0$ all $(x,y) \in A$ and $\pi(x)$ equal to the length (cost) of a shortest (least cost) directed path from s to x.

3. Solution procedure. Consider the following minimum cost flow problem (discussed in [1, Chap. III, §3]):

$$\text{minimize} \sum_A a(x,y)f(x,y) \tag{3.1}$$

subject to the constraints

$$f(x,N) - f(N,x) = \begin{cases} v, & x = s, \\ -v, & x = t, \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

$$f(x,y) \leq c(x,y), \quad (x,y) \in A, \tag{3.3}$$

$$f(x,y) \geq 0, \quad (x,y) \in A. \tag{3.4}$$

As described in [1], the labeling process can be used to solve this problem parametrically in v, for all v satisfying $0 \leq v \leq V$, where V is the maximum amount of flow from s to t. The algorithm generates a finite sequence of integral flows f_p of nondecreasing amounts v_p, $p = 0,1,\ldots,P$, where $v_0 = 0$, each f_p is a least cost flow from s to t of amount v_p, and $v_P = V$. (The complete least cost profile $a(v)$, for $0 \leq v \leq V$, can then be obtained from the sequence of points (v_p,a_p), $p = 0,1,\ldots,P$, where $a_p = \sum_A a(x,y)f_p(x,y)$, by joining distinct adjacent points of this sequence in the (v,a)-plane with line segments. The resulting function is piecewise linear and convex.) In the course of the algorithm, other numbers are generated: at stage p of the computation, certain nonnegative node integers $\pi_p(x)$ and resulting nonnegative arc integers $\gamma_p(x,y) = \max(0,\pi_p(y) - \pi_p(x) - a(x,y))$ are obtained. (Here $\pi_0(x) = 0$, $x \in N$, and hence $\gamma_0(x,y) = 0$, $(x,y) \in A$.)

These numbers satisfy the optimality properties (for the stage p problem)

(3.5) $$\pi_p(t) = p, \quad \pi_p(s) = 0,$$

(3.6) $$\pi_p(y) - \pi_p(x) < a(x,y) \Rightarrow f_p(x,y) = 0,$$

(3.7) $$\pi_p(y) - \pi_p(x) > a(x,y) \Rightarrow f_p(x,y) = c(x,y).$$

(In place of (3.7), we can write

(3.7') $$\gamma_p(x,y) > 0 \Rightarrow f_p(x,y) = c(x,y).)$$

It follows that (see [1, p. 117])

(3.8) $$pv_p - \sum_A a(x,y)f_p(x,y) = \sum_A c(x,y)\gamma_p(x,y).$$

For $v_p > 0$, consider the corresponding functions f_p, π_p, and γ_p. Define

(3.9) $$b_p = \sum_A c(x,y)\gamma_p(x,y)$$

(3.10) $$\lambda_p = 1/v_p$$

(3.11) $$g_p(x,y) = \lambda_p f_p(x,y), \quad (x,y) \in A.$$

Then π_p and γ_p satisfy (2.2)-(2.4) for $b = b_p$, and g_p satisfies (2.6)-(2.8). Moreover, from (3.5) and (3.8) we have

$$p = \pi_p(t) - \pi_p(s) = \pi_p(t) \tag{3.12}$$

$$= \sum_A a(x,y)g_p(x,y) + \lambda_p b_p .$$

Thus (π_p, γ_p) and (g_p, λ_p) are optimal solutions, respectively, to the pair of dual linear programs (2.1)-(2.4), (2.5)-(2.9), corresponding to the budget $b = b_p$. From this it can be shown that solving the minimum cost flow problem (3.1)-(3.4) parametrically in v is equivalent to solving the linear program (2.1)-(2.4) parametrically in b.

We conclude with a small example illustrating the solution process for (2.1)-(2.4). Let G be the network shown in Figure 3.1 below, with the given data (c,a) recorded as ordered pairs on arcs:

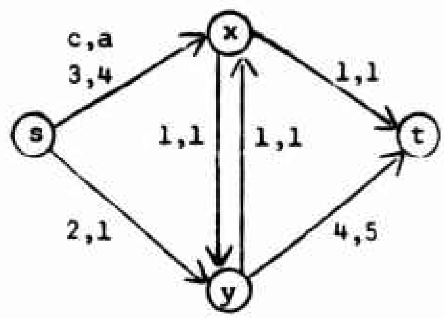
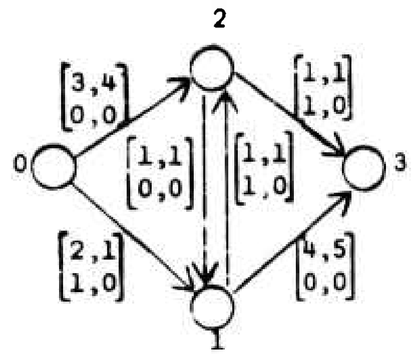


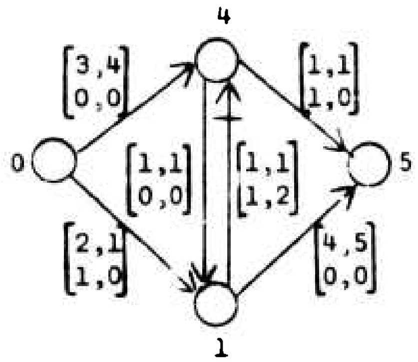
Fig. 3.1

If we go through the solution process of [1, Chap. III, §3], interpreting the function c as the arc-capacity function and the function a as the arc-cost function, the following sequence of diagrams (Figure 3.2) indicates the various relevant stages (corresponding to minimum s to t directed path lengths $p = 3, 5, 6, 8, 10, 11$) of the computation. Data are recorded beside each arc in the form $\begin{bmatrix} c & a \\ f & \gamma \end{bmatrix}$; the node numbers π are recorded beside each node barred arcs are those where $\gamma > 0$.



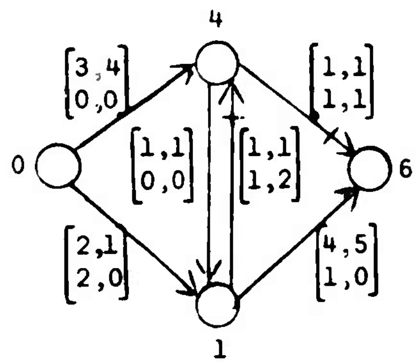
$p = 3, \quad b = 0, \quad \lambda = 1$

$a \cdot g + \lambda b = 3$



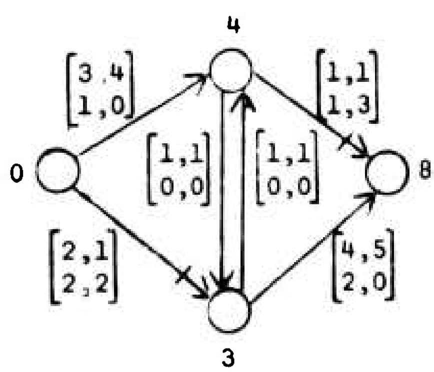
$p = 5, \quad b = 2, \quad \lambda = 1$

$a \cdot g + \lambda b = 3 + 2 = 5$



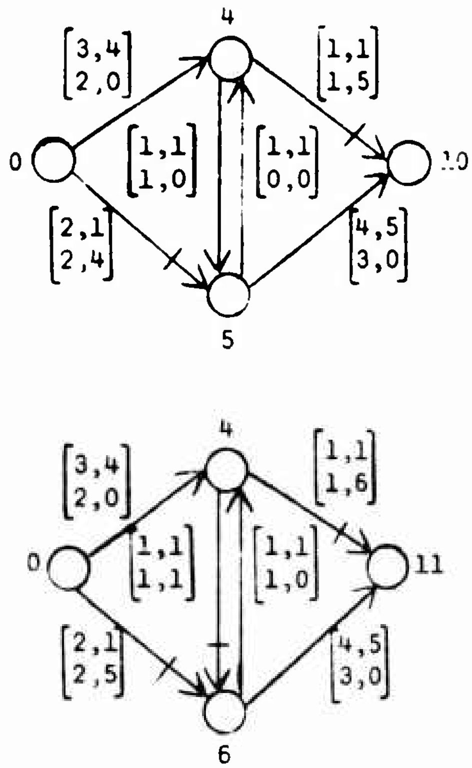
$p = 6, \quad b = 3, \quad \lambda = 1/2$

$a \cdot g + \lambda b = \frac{9 + 3}{2} = 6$



$p = 8, \quad b = 7, \quad \lambda = 1/3$

$a.g + \lambda b = \frac{17 + 7}{3} = 8$



$p = 10, \quad b = 13, \quad \lambda = 1/4$

$a \cdot g + \lambda b = \dfrac{27 + 13}{4} = 10$

$p = 11, \quad b = 17, \quad \lambda = 1/4$

$a \cdot g + \lambda b = \dfrac{27 + 17}{4} = 11$

Fig. 3.2

The last diagram in Fig. 3.2 shows a maximum flow f of amount $v = 4$, with a minimum cut $(\{s,x\},\{y,t\})$ separating s from t of capacity 4 consisting of the arcs (s,y), (x,y), (x,t). For budget $b > 17$, i.e. for least s to t directed path length $p > 11$, we would increase $\pi(y)$ and $\pi(t) = p$ by a constant amount Δ, causing $\gamma(s,y)$, $\gamma(x,y)$, and $\gamma(x,t)$ to increase by Δ. In the example, these are the only arcs on which the budget would be allocated at this stage, but this is not so in general. What is true in general is that other arcs, not in the minimum cut but on which money is being spent to increase their traversal times, would not receive any further allocation from increased budgets. (See, for instance, the example analyzed in [1, Chap. III, §3].

Budgets b intermediate between two successive ones in Figure 3.2 can be allocated by taking the appropriate convex combination of the two successive solutions. Figure 3.3 shows the minimum s to t directed path length p as a function of the budget b, obtained by joining successive (b,p) pairs of Figure 3.2 with line segments. The least path length p(b) is a concave, piecewise linear function of b, the slopes of successive pieces being equal to the successive values of $\lambda = 1/v$.

Notice the way resources are expended on the arcs (x,y) and (y,x), or, equivalently, on the undirected arc joining x and y, throughout the diagrams of Figure 3.2. For small budgets, resources are allocated to increase its length, for medium budgets no resources are allocated to this arc; but for large budgets, its length is again increased.

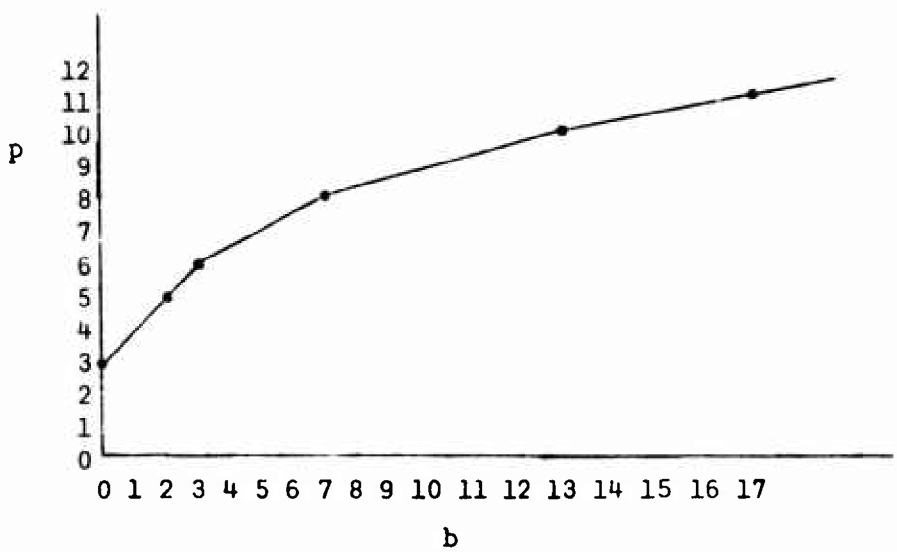


Fig. 3.3

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2. Fulkerson, D. R., "Increasing the capacity of a network: the parametric budget problem, Man. Sci. 5 (1959), 472-483.